Quantum Mechanics with Random Imaginary Scalar Potential

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Abstract

We study spectral properties of a non-Hermitian Hamiltonian describing a quantum particle propagating in a random imaginary scalar potential. Cast in the form of an effective field theory, we obtain an analytical expression for the ensemble averaged one-particle Green function from which we obtain the density of complex eigenvalues. Based on the connection between non-Hermitian quantum mechanics and the statistical mechanics of polymer chains, we determine the distribution function of a self-interacting polymer in dimensions $d > 4$.

The field of non-Hermitian quantum mechanics has, in recent years, attracted great interest. A variety of applications have been identified including the study of anomalous diffusion in random media [1], scattering in open quantum systems [2], neural networks [3], chiral symmetry breaking in quantum chromodynamics [4,5], and the statistical mechanics of flux lines in superconductors [6]. The last of these has involved the study of the quantum mechanics of a particle confined to a random impurity potential and subject to an imaginary *vector* potential. This work revealed a novel mechanism of "delocalisation" of the quantum particle, sharply contrasting with the behaviour of the Hermitian counterpart [6–12]. Here we investigate the spectral properties of a quantum particle confined to an imaginary *scalar* potential. To motivate our investigation, we apply these results to the study of the statistical mechanics of a self-interacting polymer chain.

The Hamiltonian describing a particle propagating in a random scalar potential is defined by

$$\hat{H} = \frac{\hat{\mathbf{p}}^2}{2m} + iV(\mathbf{r}), \tag{1}$$

where the potential V is drawn from a Gaussian distribution with zero mean, and correlator $\langle V(\mathbf{r})V(\mathbf{r}')\rangle_V = \gamma\delta^d(\mathbf{r}-\mathbf{r}')$. To help motivate our discussion, we will apply our analysis to the statistical mechanics of a polymer chain.

In the continuum limit, the probability distribution $Z(\mathbf{r},t)$ of the end-to-end distance $\mathbf{r}$ of a self-interacting polymer chain of length t can be expressed in the form of a path integral [13,14],

$$Z(\mathbf{r},t) = \int_{\mathbf{x}(0)=\mathbf{0}}^{\mathbf{x}(t)=\mathbf{r}} D\mathbf{x}(\tau)\exp\left\{-\int_0^t d\tau\frac{m}{2}\left(\frac{d\mathbf{x}}{d\tau}\right)^2 - \int_0^t\int_0^t d\tau' d\tau''\frac{\gamma}{2}\delta^d(\mathbf{x}(\tau')-\mathbf{x}(\tau''))\right\}. \tag{2}$$

The first term, the Wiener measure, determines the entropic contribution, while the second term represents the repulsive local contact interaction between the monomers that make up the chain. Decoupling the interaction by means of a Hubbard-Stratonovich field, the distribution function can be identified with the ensemble average of the Feynman propagator of the Hamiltonian above (1). Defining $\hat{U}(t) = \exp[-t\hat{H}]$,

$$U(\mathbf{r},t) \equiv \left\langle \mathbf{r}|\hat{U}(t)|\mathbf{0}\right\rangle = \int_{\mathbf{x}(0)=\mathbf{0}}^{\mathbf{x}(t)=\mathbf{r}} D\mathbf{x}(\tau) \exp\left\{ -\int_0^t d\tau \left[\frac{m}{2}\left(\frac{d\mathbf{x}}{d\tau}\right)^2 - iV(\mathbf{x}(\tau))\right]\right\}, \tag{3}$$

the distribution is obtained from the ensemble average $Z(\mathbf{r},t) = \langle U(\mathbf{r},t)\rangle_V$.

Applying a spectral decomposition of the complex Green function

$$\hat{g}(z) \equiv \frac{1}{z-\hat{H}} = \sum_i |R_i\rangle \frac{1}{z-z_i} \langle L_i|, \tag{4}$$

where $|R_i\rangle$ and $\langle L_i|$ denote the right and left-hand eigenfunctions of $\hat{H}$, and z_i denote the complex eigenvalues, the distribution function (2) takes the form

$$Z(\mathbf{r},t) = \frac{1}{\pi}\int d^2z \exp[-tz] \frac{\partial}{\partial z^*} \left\langle g(\mathbf{r},0;z)\right\rangle_V , \tag{5}$$

where the integration runs over the entire complex plane. Previous studies of the spectral properties of weakly non-Hermitian operators have largely (although not exclusively) focussed on properties of random matrix ensembles [15–18]. Such studies [17] have emphasised the pitfalls of a diagrammatic analysis based on a perturbative expansion of $\hat{g}$ in the random potential. The problems are revealed by representing the density of complex eigenvalues through the operator identity

$$\rho(z) \equiv \sum_i \delta^2(z-z_i) = \frac{1}{\pi}\frac{\partial}{\partial z^*} \mathrm{tr}\ \hat{g}(z), \tag{6}$$

wherein the Green function is shown to be *non-analytic* everywhere in which the density of states (DoS) is non-vanishing. To circumvent these difficulties, a representation has been introduced [17,7] in which the complex Green function is expressed through an auxiliary *Hermitian* operator,

$$\hat{G}^{-1}(\epsilon) \equiv \begin{pmatrix} \epsilon & z-\hat{H} \\ z^*-\hat{H}^\dagger & \epsilon \end{pmatrix} = \epsilon + \left(x - \frac{\hat{p}^2}{2m}\right)\sigma_1 - (y-V)\sigma_2, \tag{7}$$

where $z = x+iy$ and $\boldsymbol{\sigma}$ represent Pauli matrices. Making use of this construction, a relationship between $\hat{g}$ and the matrix Green function is straightforwardly obtained,

$$\hat{g}(z) = \lim_{\epsilon \to -i0} \hat{G}_{21}(\epsilon). \tag{8}$$

At the same time, this representation manifests an implicit *time-reversal* and *chiral* symmetry of the matrix Hamiltonian: $\sigma_3\hat{G}^{-1}(0)\sigma_3 = -\hat{G}^{-1}(0)$. Recently, this representation was successfully combined with diagrammatic perturbation theory to study the spectrum of the Fokker-Planck operator describing particles diffusing in a quenched random velocity field [19].

Cast in this form, we are able to apply standard field theoretic methods to obtain statistical properties of the single-particle Green function. Our approach is closely related to that recently developed by Efetov [20] (see also Ref. [21]) to investigate spectral properties of the random Schrödinger operator subject to an imaginary vector potential. This approach involves a generalisation of the supersymmetry method originally tailored to the description of disordered conductors [22]. Being somewhat more technical than the diagrammatic perturbation theory employed in [19], the use of the supersymmetry technique is nevertheless justified for the problem at hand. Indeed, although the diagrammatic approach reproduces the mean-field result, it does not take properly into account the existence of the massless Goldstone modes (see below), and therefore can not be safely used if one wants to go beyond the saddle-point approximation.

The analysis begins by expressing the matrix Green function as a functional integral over 8-component supervector fields $G_{\alpha\beta}(\mathbf{0},\mathbf{r}) = i\langle \mathrm{tr}[R^{-1}\Psi(\mathbf{0}) \otimes \bar{\Psi}(\mathbf{r})\sigma_3 R\Sigma^{\beta\alpha}]\rangle_\Psi/4$, where

$$\langle \cdots \rangle_\Psi = \int D[\bar{\Psi},\Psi]\,(\cdots)\exp\left\{-\frac{i}{2}\int \bar{\Psi}\left[i\left(x-\frac{\hat{p}^2}{2m}\right)\sigma_2 + (V-y) - i0\sigma_3\right]\Psi\right\}, \qquad (9)$$

$R = \exp[-i\pi\sigma_1/4]$, and the infinitesimal imaginary part ensures convergence. Here we adopt a standard notation [22] in which the fields Ψ, $\bar{\Psi}$ subdivide into a time-reversal (TR), a Fermion-Boson (FB), and a "spinor" or matrix sector. $\Sigma^{\beta\alpha}$ is a 2×2 matrix which projects on to the $\alpha\beta$ components in the spinor space.

Expressed in this form, an ensemble average over the random impurity potential generates a quartic interaction of the fields which can be decoupled by the introduction of 8×8 component supermatrix fields Q. Taking our notation from disordered conductors and defining $\tau^{-1}(x) = 2\pi\gamma\nu(x)$, where $\nu(x)$ is the unperturbed DoS, we obtain

$$\left\langle \exp\left[-\frac{i}{2}\int \bar{\Psi} V \Psi\right]\right\rangle_V = \int DQ \exp\left[\frac{1}{4\tau}\int\left(\frac{\pi\nu}{2}\mathrm{str}Q^2 - \bar{\Psi}Q\Psi\right)\right], \tag{10}$$

where $\mathrm{str}M = M_{\mathrm{FF}} - M_{\mathrm{BB}}$ represents the trace operation for supermatrices. The supermatrix fields Q have an algebraic structure which reflects that of the dyadic product $\Psi \otimes \bar{\Psi}$. Integrating over the superfields Ψ, we obtain $\langle G_{\alpha\beta}(\mathbf{0},\mathbf{r})\rangle_V = -\langle \mathrm{tr}[R^{-1}\mathcal{G}(\mathbf{0},\mathbf{r})\sigma_3 R\Sigma^{\beta\alpha}]\rangle_Q/4$ with

$$\langle\cdots\rangle_Q = \int DQ(\cdots)\exp\left[\int \mathrm{str}\left(\frac{\pi\nu}{8\tau}Q^2 - \frac{1}{2}\ln\hat{\mathcal{G}}^{-1}\right)\right], \tag{11}$$

where the supermatrix Green function is defined by

$$\hat{\mathcal{G}}^{-1} = i\left(x - \frac{\hat{p}^2}{2m}\right)\sigma_2 - y - \frac{i}{2\tau}Q. \tag{12}$$

Further progress is possible only within a saddle-point approximation, which is controlled in the limit of weak disorder $[y, 1/\tau] \ll x$. Minimising the action (11) with respect to variations in Q, we obtain the saddle-point equation

$$Q(\mathbf{r}) = -\frac{i}{\pi\nu}\mathcal{G}(\mathbf{r},\mathbf{r}). \tag{13}$$

The saddle-point solution is found from the ansatz that Q is homogeneous in space, and diagonal in the TR and FB sector. Applying the parametrisation $Q = q_0 + \mathbf{q}\cdot\boldsymbol{\sigma}$, we obtain two solutions [23],

$$q_0 = \begin{cases} iy\tau \\ i\mathrm{sgn}(y) \end{cases}, \qquad q_1 = q_2 = 0, \qquad q_3 = \begin{cases} (1-y^2\tau^2)^{1/2} \\ 0. \end{cases} \tag{14}$$

Moreover, invariance of the Eq. (13) under rotations $Q \to TQT^{-1}$ where $[T,\sigma_2] = 0$, shows that the first solution of the saddle-point equation spans a degenerate manifold, Class CI in the classification of Ref. [24]. Below, we will find that this solution corresponds to the non-analytic part of the Green function with a non-zero DoS. The second (non-degenerate) solution yields the analytic part of the Green function, and does not contribute to the DoS. The region in the complex plane where the first solution is stable defines the support of the spectrum. Straightforward stability analysis shows that the boundary of the spectrum is determined by the equation $y = 1/\tau(x)$.

Expanding the action in slow fluctuations $T(\mathbf{r})$ around the saddle-point solution we obtain the low energy effective action

$$S[Q] = -\frac{\pi\nu}{8}\int d\mathbf{r} D(y)\ \mathrm{str}\left(\partial Q\right)^2, \tag{15}$$

where $Q = T\sigma_3 T^{-1}$, and $D(y) = (1 - y^2\tau^2)D_0$, with $D_0 = 2x\tau(x)/md$, denotes the y-dependent classical diffusion constant. Thus, in contrast to a *real* random impurity potential, modes of density relaxation of the matrix Hamiltonian are controlled by massless Goldstone modes of a supersymmetric non-linear σ-model of symmetry class CI with a diffusion constant which depends explicitly on y.

Expanding the supermatrix Green function,

$$\mathcal{G}(\mathbf{0},\mathbf{r}) = -i\pi\nu(x) f_d(\mathbf{r})\left[q_0 + q_3 Q(\mathbf{r})\right], \tag{16}$$

where $f_d(\mathbf{r}) = \mathrm{Im}G_0^-(\mathbf{0},\mathbf{r})/\mathrm{Im}G_0^-(\mathbf{0},\mathbf{0})$ and $G_0^- = (x - \mathbf{p}^2/2m - i/2\tau)^{-1}$, we obtain

$$\langle G_{\alpha\beta}(\mathbf{0},\mathbf{r})\rangle = i\frac{\pi\nu}{4} f_d(\mathbf{r})\int DQ\ \mathrm{str}\left[R^{-1}\left(q_0 + Q(\mathbf{r})\right)\sigma_3 R\Sigma^{\beta\alpha}\right]\ e^{-S[Q]}. \tag{17}$$

Applied to the DoS, the projection matrix takes the form $\Sigma = (\sigma_1 + i\sigma_2)/2$. In this case the supermatrix degrees of freedom of Q are decoupled from the source, and the DoS is specified simply by the mean-field result

$$\rho(z) = \begin{cases} (4\pi\gamma)^{-1}, & |y| < 1/\tau(x), \\ 0 & |y| > 1/\tau(x), \end{cases} \tag{18}$$

satisfying the sum rule $\int dy\rho(z) = \nu(x)$. This result compares with that obtained for the corresponding random matrix ensemble [18], and contrasts that obtained for an imaginary vector potential [20]. In particular, since the DoS source does not couple to the effective action, the mean field estimate is unchanged by integration over Q. As a result, the complex eigenvalue density remains non-singular at $y = 0$.

We have computed numerically the eigenvalues of the two-dimensional lattice version of the Hamiltonian (1). Theory and simulation are compared in fig. 1, for 50 realisations of a 32×32 lattice with $\gamma = 1$. The solid line represents the boundary of the spectrum as

calculated in the saddle-point approximation, $y = \pm 1/\tau(x) = 2\pi\nu(x)\gamma$, $-4 < x < 4$, where $\nu(x)$ is the exact DoS of the clean two-dimensional tight-binding model. The results show good agreement between theory and numerics, with deviations becoming significant only near the edges of the band, and in the vicinity of the band center. (Note that, in contrast to the Hermitian disordered Hamiltonian, the imaginary scalar potential *raises* the energy of the low lying states, an effect easily understood within the framework of second order perturbation theory. However, the localisation properties of such "Lifshitz tail" states, as well as their sensitivity to optimal fluctuations of the impurity potential is a subtle question which lies beyond the scope of the present investigation. Secondly, a sublattice symmetry specific to the tight-binding Hamiltonian induces a reflection symmetry of the spectrum around $x = 0$. This additional symmetry, which parallels that found with the imaginary vector potential, leads to the accumulation of a finite fraction of states along $x = 0$, a phenomenon which has no counterpart in the continuum model.) We believe that the small deviations of numerics and theory away from the band edge and band centre can be ascribed to the influence of massive fluctuations of the matrix fields, i.e. those which violate the symmetry $[T(\mathbf{r}), \sigma_2] = 0$.

Turning to the self-interacting polymer chain discussed above, Eq. (16) can be used to obtain the probability distribution $Z(\mathbf{r}, t)$. However, to do so, we should recall that the saddle-point approximation is justified in the limit $x\tau \gg 1$ which, setting $m = 1$, translates to the following condition on the contact interaction: $\gamma x^{(d-4)/2} \ll 1$. The latter is satisfied as $x \to 0$ in dimensions higher than four, and $x \to \infty$ in dimensions lower than four. Equivalently, applied to the polymer chain, the analysis above applies at time scales $t \gg \gamma^{2/(d-4)}$ for $d > 4$, and $t \ll (1/\gamma)^{2/(4-d)}$ for $d < 4$ defining the upper critical dimension as $d_c = 4$ [14]. Evaluation of Eq. (5) for $d > 4$ gives the power spectrum

$$Z(\mathbf{p}, t) = \frac{\sin(\omega(\mathbf{p})t)}{\omega(\mathbf{p})t} \exp[-p^2 t], \qquad \omega(\mathbf{p}) = \pi\gamma|\mathbf{p}|^{d-2}. \tag{19}$$

This result parallels that obtained by Chalker and Wang [19] for the time evolution of the particle density of the random Fokker-Planck operator. In particular, in the limit

$t \to \infty$ it recovers diffusive behaviour $\langle r^2 \rangle \sim t$, while at short times, when the saddle-point approximation is uncontrolled, we have $\langle r^2 \rangle \sim (\gamma t)^{2/(d-2)}$. The long-time behaviour for $d < 4$ is strongly affected by low-energy states which, being sensitive to optimum fluctuations of the random impurity potential, is not accessible within the present framework.

In conclusion, applying conventional field theoretic methods, we have shown that spectral properties of the non-Hermitian Hamiltonian describing a particle propagating in an imaginary scalar potential are governed by a supersymmetric non-linear σ-model (of symmetry class CI [24]). The latter has been used to obtain the complex eigenvalue density. As an application of these results, we have obtained the power spectrum of the probability distribution function of a self-interacting polymer chain. Although the analysis developed here has focussed on one-particle properties, an extension of the present approach to treat fluctuation phenomena is, at least formally, straightforward (see, for example, Ref. [25]).

Acknowledgements: We are grateful to Alexander Altland, John Chalker, Igor Lerner, Charles Offer, and Damian Taras-Semchuk for illuminating discussions. One of us (A.V.I.) would like to acknowledge the financial support of Trinity College, Cambridge.

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FIGURES

FIG. 1. Complex eigenvalues taken from 50 realisations of a 32×32 square lattice tight-binding model with $\gamma = 1.0$. The solid line is $y = \pm 1/\tau(x) = 2\pi\nu(x)\gamma$.